Quantum field theory of treasury bonds

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The Heath-Jarrow-Morton (HJM) formulation of treasury bonds in terms of forward rates is recast as a problem in path integration. The HJM model is generalized to the case where all the forward rates are allowed to fluctuate independently. The resulting theory is shown to be a two-dimensional Gaussian quantum field theory. The no arbitrage condition is obtained and a functional integral derivation is given for the price of a futures and an options contract.

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I. INTRODUCTION

Stochastic calculus is the most widely used mathematical formalism for modeling financial instruments [1], followed by the use of partial differential equations [2]. The Feynman path integral is a formalism based on functional integration and is widely used in theoretical physics to model quantum (random) phenomenon [3]; it is also ideally suited for studying stochastic processes arising in finance. In Refs. [4,5] techniques from physics were applied to the study of finance. In Ref. [6] the problem of the pricing of stock options with stochastic volatility was studied analytically and numerically using the formalism of path integration.

In this paper, the path integral approach is applied to the field of interest rates, also called the forward rates, as embodied in the modeling of Treasury bonds. The complexity of this problem is far greater than that encountered in the study of stocks and their derivatives; the reason being that a stock at a given instant in time is described by only one stochastic variable undergoing random evolution whereas in the case of the interest rates it is the entire yield curve which is randomly evolving and requires infinitely many independent variables for its description. The theory of quantum fields [7] has been developed precisely to study problems involving infinitely many variables and so we are naturally led to the techniques of quantum field theory in the study of the interest yield curve.

Treating all the forward rates as independent random variables has also been studied in [8–11], where a stochastic partial differential equation in infinitely many variables is written. The approach based on quantum field theory proposed in this paper is in some sense complimentary to the approach based on stochastic partial differential equations since the expressions for all financial instruments are formally given as a functional integral. One advantage of the approach based on quantum field theory is that it offers a different perspective on financial processes, offers a variety of computational algorithms, and nonlinearities in the forward rates as well as its stochastic volatility can be incorporated in a fairly straightforward manner.

The Heath-Jarrow-Morton (HJM) model [12] is taken as the starting point of this paper. In Sec. II the HJM model is reexpressed in terms of a path integral, and the condition of no arbitrage is rederived in this formalism. For readers unfamiliar with the financial concept of arbitrage, a brief discussion is given in Sec. III. To make the formalism more transparent and accessible to readers not familiar with path integration, the well-known results of the HJM model for the price of futures of zero-coupon bonds as well as the price of a European call option and a cap for a zero-coupon bond is derived in Sec. IV. Another, and more important reason, for these rederivations is that the prices of these derivatives are expressed in a form that can be directly generalized to the case when we model the evolution of the forward rates using quantum field theory.

In Sec. VI, a brief review of quantum mechanics and quantum field theory is provided for readers from fields other than physics.

In Sec. VII, the HJM model is generalized to the case with independent fluctuations of all the forward rates; the theory is seen to consist of a free (Gaussian) two-dimensional quantum field theory. The generalized model has a new parameter which determines how strongly it deviates from the HJM model. The condition of no arbitrage is derived for the generalized model.

In Sec. X, the formulas for the prices of futures and options of zero-coupon bonds are obtained explicitly for the Gaussian quantum field theory.

In Sec. XI some conclusions are discussed as well as possible future directions of research.

II. PATH INTEGRAL FORMULATION OF THE HJM MODEL

Bonds are financial instruments of debt that are issued by governments and corporations to raise money from the capitals market. Bonds have a predetermined (deterministic) cash flow; a treasury bond is an instrument for which there is no risk of default in receiving the payments, whereas for corporate bonds there is, in principle, such a risk. A treasury *zero-coupon bond* is a risk-free financial instrument that has a single cash flow consisting of a fixed payoff of say $1 at some future time T; its price at time $t<T$ is denoted by $P(t,T)$, with $P(T,T)=1$.

A Treasury *coupon bond* $\mathcal{B}(t,T)$ has a series of predetermined cash flows that consist of coupons worth c_i paid out at increasing times T_i's, and with the principal worth L being paid at time T. $\mathcal{B}(t,T)$ is given in terms of the zero-coupon bonds by [13]

$$\mathcal{B}(t,T)=\sum_{i=1}^{K} c_i P(t,T_i)+LP(t,T). \quad (2.1)$$

From above we see that a coupon bond is equivalent to a portfolio of zero-coupon bonds. Hence, if we model the behavior of zero-coupon bonds, we automatically have a model for coupon bonds as well

Consider the forward rate $f(t,x)$, which stands for the spot (overnight) interest rate at future time x for a contract entered into at time $t<x$. The price of a zero-coupon bond with the value of $1 at maturity is given by

$$P(t,T)=\exp\left\{-\int_t^T dx f(t,x)\right\}. \quad (2.2)$$

Note from its definition, the spot rate for an overnight loan at some (future) time t is $r(t)$ and is given by

$$r(t)=f(t,t). \quad (2.3)$$

The forward rate is a stochastic variable. In the K-factor HJM-model [12–14] the time evolution of the forward rates is given by

$$\frac{\partial f}{\partial t}(t,x)=\alpha(t,x)+\sum_{i=1}^{K} \sigma_i(t,x)W_i(t), \quad (2.4)$$

where $\alpha(t,x)$ is the drift velocity term and $\sigma_i(t,x)$ are the deterministic volatility functions of the forward rates. From Eq. (2.4) we have

$$f(t,x)=f(t_0,x)+\int_{t_0}^{t} dt'\,\alpha(t',x) + \int_{t_0}^{t} dt' \sum_{i=1}^{K} \sigma_i(t',x)W_i(t'). \quad (2.5)$$

The initial forward rate $f(t_0,x)$ is determined from the market, and so are the volatility functions $\sigma_i(t,x)$.

For every value of time t, the stochastic variable $W_i(t), i=1,2,\ldots,K$ is an independent Gaussian random variable — also called white noise — given by

$$E(W_i(t)W_j(t'))=\delta_{ij}\delta(t-t'). \quad (2.6)$$

Note that the forward rates $f(t,x)$ are driven by random variables $W_i(t)$ that give the same random ''shock'' to all the forward rates; the volatility function $\sigma(t,x)$ weighs this ''shock'' differently for each time t and each x. It is precisely this feature that we will generalize later such that $f(t,x)$ is taken to be an *independent* random variable for *each* x and *each* t.

To write the probability measure for $W_i(t)$, note that t takes values in a finite interval depending on the problem of interest; we discretize $t\to m\epsilon$, with $m=1,2,\ldots,M$, and with $W_i(t)\to W_i(m)$. The probability measure is given by

$$\mathcal{P}[W]=\prod_{m=1}^{M}\prod_{i=1}^{K} \exp\left[-\frac{\epsilon}{2}\sum_{i=1}^{K} W_i^2(m)\right], \quad (2.7)$$

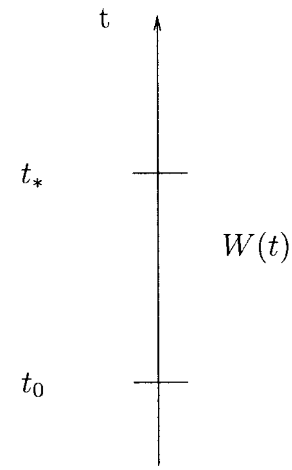

FIG. 1. Independent $W(t)$ random variables in the HJM model.

$$\int dW=\prod_{m=1}^{M}\prod_{i=1}^{K}\sqrt{\frac{\epsilon}{2\pi}}\int_{-\infty}^{+\infty} dW_i(m). \quad (2.8)$$

For notational simplicity we take the limit of $\epsilon\to 0$; note that for purposes of rigor, the continuum notation is simply a short hand for taking the continuum limit of the discrete multiple integrals given above. We have, for $t_1<t<t_2$,

$$\mathcal{P}[W,t_1,t_2]\to e^{S_0}, \quad (2.9)$$

$$S_0\equiv S_0[W,t_1,t_2]=-\frac{1}{2}\sum_{i=1}^{K}\int_{t_1}^{t_2} dt W_i^2(t), \quad (2.10)$$

$$\int dW\to\int DW. \quad (2.11)$$

The ''action'' functional S_0 is ultralocal with all the variables being decoupled; generically, $\int DW$ stands for the (path) integration over all the random variables $W(t)$ that appear in the problem. The integration variables $W_i(t)$ are shown in Fig. 1, where each point t in the interval $t\in[t_0,t_*]$ represents an independent random variable $W_i(t)$.

A path integral approach to the HJM model has been discussed in [15]; the action derived is different than the one given above since a different set of variables is used resulting in an action involving time derivatives.

A formula that we will repeatedly need is the generating functional for W given by the path integral

$$Z[j,t_1,t_2]=\int DW\exp\left\{\sum_{i=1}^{K}\int_{t_1}^{t_2} dt j_i(t)W_i(t)\right\}e^{S_0[W,t_1,t_2]} =\exp\left[\frac{1}{2}\sum_{i=1}^{K}\int_{t_1}^{t_2} dt j_i^2(t)\right]. \quad (2.12)$$

III. EFFICIENT MARKET, ARBITRAGE AND MARTINGALE

Arbitrage— an idea that is central to finance — is a term for gaining an instantaneous risk-free (guaranteed) profit by simultaneously entering into two or more financial transactions — be it in the same market or in two or more different markets. Since one has risk-free instruments such as treasury bonds, arbitrage means obtaining risk-free returns *above* the risk-less returns that one can get from the market.

For example, suppose that at some instant the share of a company is traded at value US$1 in the New York stock exchange, and at value S$1.8 in Singapore, with the currency conversion being US$1=S$1.7. A broker can simultaneously buy 100 shares in New York and sell 100 shares in Singapore making a riskless profit of S$10. Transaction costs tend to cancel out arbitrage opportunities for small traders, but for big brokerage houses — which have virtually zero transaction cost — arbitrage is a major source of profits. One can also see that the price of the share in Singapore will tend to move to a value close to S$1.7 due to the selling of shares by the arbitrageurs.

An *efficient market* is one in which there are no arbitrage opportunities. In an intuitive manner of speaking, no arbitrage means that ''there is no free lunch.'' Arbitrage is the mechanism by which the capitals market effectively functions as an efficient market, and determines the equilibrium ''correct'' price of any financial instrument.

An important result of theoretical finance is the following: for the price of any financial instrument to be free from any possibility of arbitrage — as is the case for an efficient market — it is necessary to have a Martingale measure for the evolution of the financial instrument [13,16]. The concept of a Martingale in probability theory is the mathematical formulation of the concept of a fair game, and is equivalent, in finance, to the principle of an efficient market.

Suppose a gambler is playing a game of tossing a fair coin, represented by a discrete random variable Y with two equally likely possible outcomes ± 1; that is, $P(Y=1)=P(Y=-1)=\frac{1}{2}$. Let X_n represent the amount of cash that the gambler has after n identical throws. That is, $X_n=\Sigma_{i=1}^{n}Y_i$, where Y_i's are random variables all identical to Y; let x_n denote some specific outcome of random variable X_n. The Martingale condition states that the expected value of the cash that the gambler has on the $(n+1)$th throw must be equal to the cash that he is holding at the nth throw. Or in equations

$$E[X_{n+1}|X_1=x_1,X_2=x_2,\ldots,X_n=x_n]=x_n,\quad \text{Martingale condition.} \tag{3.1}$$

In other words, in a fair game, the gambler — on the average — simply leaves the casino with the cash that he came in with.

A. The condition of no arbitrage

The Martingale condition applied to the evolution of the forward rates states the following. Suppose that a zero-coupon bond that matures at time T has a price $P(t_0,T)$ for some $t_0<T$, and a price, before expiry, of $P(t_*,T)$ at time $t_*>t_0$. The Martingale condition states that the price of the bond at t_*, evolved backward to time t_0 and continuously discounted by the risk-free spot rate $r(t)$, must be equal to the price of the bond at time t_0.

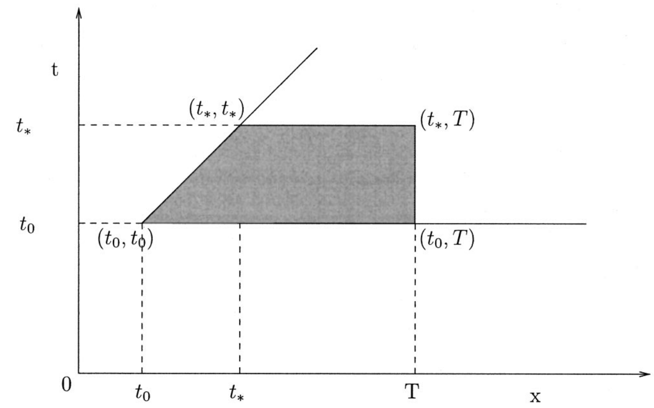

FIG. 2. Trapezoidal domain $\mathcal{T}$ for no arbitrage.

We now show that the no arbitrage condition for the forward rates implies that the drift velocity of the forward rates, namely, $\alpha(t,x)$ is not an independent quantity, but rather has to be made a function of its volatility.

The Martingale condition implies for the zero-coupon bond, analogous to Eq. (3.1) yields

$$P(t_0,T)=E_{[t_0,t_*]}\left[\exp\left\{-\int_{t_0}^{t_*}r(t)dt\right\}P(t_*,T)\right], \tag{3.2}$$

where the notation $E_{[t_0,t_*]}[S]$ denotes the average of the stochastic variable S over the time interval(t_0,t_*). From Eqs. (2.2), (2.5), and (3.2) we have that

$$P(t_0,T)=P(t_0,T)e^{-\int_{\mathcal{T}}\alpha(t,x)}\int DW \times\exp\left\{-\sum_i \int_{\mathcal{T}}\sigma_i(t,x)W_i(t)e^{S_0[W]}\right\}, \tag{3.3}$$

where the *trapezoidal* domain $\mathcal{T}$ is given in Fig. 2 and

$$\int_{\mathcal{T}}\equiv\int_{t_0}^{t_*}dt\int_t^T dx. \tag{3.4}$$

On performing the W integrations we obtain from Eq. (3.3)

$$\exp\left\{\int_{\mathcal{T}}\alpha(t,x)\right\}=\exp\left\{\frac{1}{2}\int_{t_0}^{t_*}dt\sum_i\left[\int_t^T dx\sigma_i(t,x)\right]^2\right\}. \tag{3.5}$$

Dropping the integration over t we obtain [13]

$$\int_t^T dx\alpha(t,x)=\frac{1}{2}\sum_{i=1}^{K}\left[\int_t^T dx\sigma_i(t,x)\right]^2, \tag{3.6}$$

or equivalently

$$\alpha(t,x)=\sum_{i=1}^{n}\sigma_i(t,x)\int_t^x dy\,\sigma_i(t,y),$$

$$\text{condition for no arbitrage.} \tag{3.7}$$

We see that, as expected, the Martingale condition leads to the well-known [13] no-arbitrage condition that expresses the drift velocity of the forward rates in terms of its volatility.

Consider the two-factor HJM model with volatilities given by

$$\sigma_1(t,x)=\sigma_1;\sigma_2(t,x)=\sigma_2 e^{-\lambda(x-t)}. \tag{3.8}$$

The no-arbitrage condition given in Eq. (3.7) for this case yields

$$\alpha(t,x)=\sigma_1^2(x-t)+\frac{\sigma_2^2}{\lambda}e^{-\lambda(x-t)}(1-e^{-\lambda(x-t)}). \tag{3.9}$$

B. Spot rate models for the forward rates

There are two different approaches to the modeling of forward interest rates. The HJM approach takes the initial forward rate curve $f(t_0,x)$ as input, and models the future evolution of the entire curve. In the spot rate approach [14,17,18], only the spot rate $r(t)=f(t,t)$ is studied, and the entire forward rate curve $f(t,x)$ is *derived* from the spot rate. Typically, a spot rate model is determined by the following stochastic differential equation

$$\frac{dr(t)}{dt}=a(r)+b(r)W(t) \tag{3.10}$$

with the initial condition

$$r(t_0)=r_0, \tag{3.11}$$

where a,b are chosen to have various functional forms, and $W(t)$ is white noise.

All spot rate models by definition respect the condition of no arbitrage. The Martingale condition given in Eq. (3.2), instead of yielding the no-arbitrage condition as is the case for models of the forward rates, is the *defining equation* for the forward rates in terms of the spot rate.

To determine the forward rates $f(t_0,x)$, or equivalently, the price of the zero coupon bond $P(t_0,T)$, from the spot rate $r(t)$, set $t_*=T$ in Eq. (3.2); this changes the trapezoidal domain $\mathcal{T}$, given in Fig. (2), to a (right isosceles) *triangular* domain Δ given in Fig. 3 —which is also the largest domain in the study of treasury bonds expiring at time T. We hence have, from Eq. (3.2) and using $P(T,T)=1$, the following:

$$P(t_0,T)\equiv E_{[t_0,T]}\left[\exp\left\{-\int_{t_0}^{T}r(t)dt\right\}\right]. \tag{3.12}$$

If the functions a and b in Eq. (3.10) are independent of the spot rate $r(t)$, we have a class of models called the *affine models* with the result that

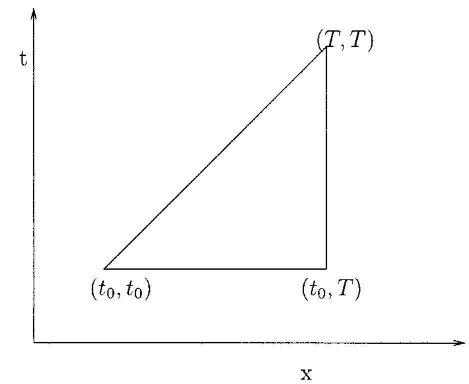

FIG. 3. Domain Δ.

$$P(t_0,T)=A(t_0,T)e^{-B(t_0,T)r_0}, \tag{3.13}$$

where $A,B>0$. One can easily derive from the expression above the forward rate $f(t_0,x)$ for $t_0<x<T$.

IV. FUTURES PRICING IN THE HJM MODEL

The future and forward contracts on a zero-coupon bond are instruments that are traded in the capitals market. The forward and future price of $P(t,T)$, namely, $F(t_0,t_*,T)$ and $\mathcal{F}(t_0,t_*,T)$, respectively, is the price fixed at time $t_0<t_*$ for having a zero-coupon bond — maturing at T — to be delivered to the buyer at time $t_*<T$.

The difference in the two instruments is that for a forward contract there is only a single cash flow at t_*: the expiry date of the contract . For a futures contract on the other hand there is a continuous cash flow from time t_0 to t_* such that all variations in the price of $P(t+dt,T)$ away from $P(t,T)$, for $t_0<t<t_*$, are settled continuously between the buyer and the seller, with a final payment of $P(t_*,T)$ at time t_*. If the time evolution of $P(t,T)$ was deterministic, it is easy to see that the forward and futures price would be equal.

It can be shown that the price of the futures $\mathcal{F}$ is given by [13]

$$\mathcal{F}(t_0,t_*,T)=E_{[t_0,t_*]}[P(t_*,T)]. \tag{4.1}$$

From Eqs. (2.5) and (2.9) we have

$$\mathcal{F}(t_0,t_*,T)=\int DW\exp\left[-\int_{t_*}^{T}dxf(t_*,x)\right]\mathcal{P}[W,t_0,t_*] \tag{4.2}$$

$$=F(t_0,t_*,T)\exp\Omega_{\mathcal{F}}, \tag{4.3}$$

where the forward price for the same contract is given by [13]

$$F(t_0,t_*,T)=\frac{P(t_0,T)}{P(t_0,t_*)}. \tag{4.4}$$

The trapezoidal domain $\mathcal{T}$ splits into a triangle and a rectangle shown in Fig. 4 and yields

$$\mathcal{T}=\Delta_0\oplus\mathcal{R}. \tag{4.5}$$

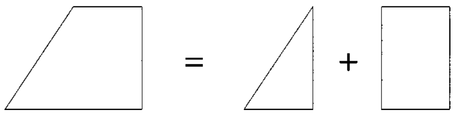

FIG. 4. Domain $\mathcal{T}$= domain $\Delta_0 \oplus$ domain $\mathcal{R}$.

The futures price is defined over the *rectangular* domain $\mathcal{R}$ given in Fig. 5 and

$$\int_{\mathcal{R}} \equiv \int_{t_0}^{t_*} dt \int_{t_*}^{T} dx. \tag{4.6}$$

We have

$$\exp\Omega_{\mathcal{F}} = e^{\Omega} \exp\left[-\int_{\mathcal{R}} \alpha(t,x) \right] \tag{4.7}$$

with

$$e^{\Omega} = \int DW \exp\left[-\sum_{i=1}^{K} \int_{\mathcal{R}} \sigma_i(t,x) W_i(t) \right] e^{S_0} \tag{4.8}$$

$$= \exp\left\{ \frac{1}{2} \sum_{i=1}^{K} \int_{t_0}^{t_*} dt \left[\int_{t_*}^{T} dx\, \sigma_i(t,x) \right]^2 \right\}, \tag{4.9}$$

where Eq. (4.9) has been obtained by performing the path integration over the W variables using Eq. (2.12).

Using the no-arbitrage condition given in Eq. (3.7), and after some simplifications, we obtain from Eq. (4.7) that

$$\Omega_{\mathcal{F}}(t_0,t_*,T) = -\sum_{i=1}^{K} \int_{t_0}^{t_*} dt \int_{t}^{t_*} dx\, \sigma_i(t,x) \int_{t_*}^{T} dx'\, \sigma_i(t,x'). \tag{4.10}$$

As is expected, the future and forward prices of the zero-coupon bond are equal if the volatility is zero, that is, the evolution of the zero-coupon bond is deterministic.

Consider the two-factor HJM model with volatilities given in Eq. (3.8). Equation (4.10) yields

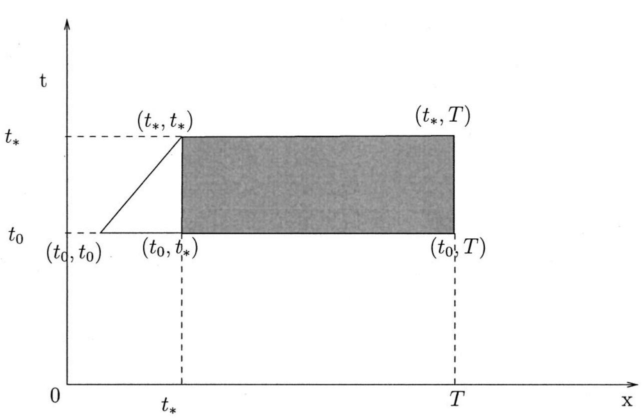

FIG. 5. Domain $\mathcal{R}$ is shaded above.

$$\Omega_{\mathcal{F}}(t_0,t_*,T) = -\sigma_1^2 (T-t_*)(t_*-t_0)^2 - \frac{\sigma_2^2}{2\lambda^3}(1-e^{-\lambda(T-t_*)})(1-e^{-\lambda(t_*-t_0)})^2, \tag{4.11}$$

which is the result given in [1].

V. OPTION AND CAP PRICING IN THE HJM MODEL

Suppose we need the price, at time t_0, of a derivative instrument of a zero-coupon bond $P(t,T)$ for a contract that expires at $t_* < T$. For concreteness we study the price of a European call option on a zero-coupon bond [13,14], namely, $C(t_0,t_*,T,K)$; the option has a strike price of K and exercise time at $t_* > t_0$.

The final value of the option at $t_0 = t_*$ is, as required by the contract, given by

$$C(t_*,t_*,T,K) = [P(t_*,T) - K]_+ \tag{5.1}$$

$$\equiv [P(t_*,T) - K]\,\theta(P(t_*,T) - K), \tag{5.2}$$

where the step function is defined by

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{5.3}$$

For $t_0 < t_*$ the price of C given by

$$C(t_0,t_*,T,K) = E_{[t_0,t_*]}\left\{ \exp\left[-\int_{t_0}^{t_*} dt f(t,t) \right] \times [P(t_*,T) - K]_+ \right\}. \tag{5.4}$$

The expectation value in Eq. (5.4) is taken by evolving the payoff function $[P(t_*,T)-K]_+$ backward from t_* to t_0, discounted by stochastic spot rate $r(t) = f(t,t)$.

Using the identity

$$\delta(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp\, e^{ipz}, \tag{5.5}$$

we can rewrite Eq. (5.4)

$$C(t_0,t_*,T,K) = \int_{-\infty}^{+\infty} dG\, \Psi(G,t_*,T)(e^{G} - K)_+ , \tag{5.6}$$

where

$$\Psi(G,t_*,T)=E_{[t_0,t_*]}\left[\exp\left\{-\int_{t_0}^{t_*}dtf(t,t)\right\}\times\delta(\ln\{P(t_*,T)-e^G\})\right], \tag{5.7}$$

$$=\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{\Lambda}e^{ip(G+\Lambda_0)}, \tag{5.8}$$

$$\Lambda_0=\ln F(t_0,t_*,T). \tag{5.9}$$

Using Eqs. (2.9) and (2.5), we have the following:

$$e^{\Lambda}=\exp\left[-\int_{\Delta_0}\alpha(t,x)+ip\int_{\mathcal{R}}\alpha(t,x)\right]\int DW \times\exp\left[-\int_{\Delta_0}\sigma_i(t,x)W_i(t) +ip\sum_i^K\int_{\mathcal{R}}\sigma_i(t,x)W_i(t)\right]e^{S_0}. \tag{5.10}$$

Note the subtle interplay of the subdomains Δ_0 and $\mathcal{R}$ in determining the price of the option. Using Eq. (2.12) to perform the integrations over W yields, after considerable simplifications and using the no-arbitrage condition given by Eq. (3.6), we obtain

$$\Lambda=-\frac{q^2}{2}(p^2+ip) \tag{5.11}$$

with

$$q^2=\sum_{i=1}^K\int_{t_0}^{t_*}dt\left[\int_{t_*}^{T}dx\sigma_i(t,x)\right]^2. \tag{5.12}$$

To obtain Eq. (5.11) we have used the identity

$$\int_{t_0}^{t_*}dt\left[\int_{t_*}^{T}dx\alpha(t,x)-\sum_{i=1}^K\int_t^{t_*}dx\sigma_i(t,x)\int_{t_*}^{T}dy\sigma_i(t,y)\right] =\frac{1}{2}q^2. \tag{5.13}$$

Performing the Gaussian integration in Eq. (5.8) yields

$$\Psi(G,t_*,T) =\sqrt{\frac{1}{2\pi q^2}}\exp-\frac{1}{2q^2}\left\{G+\int_{t_*}^{T}dxf(t_0,x)-\frac{q^2}{2}\right\}^2. \tag{5.14}$$

Hence from above and Eq. (5.6) we recover the well-known result [19,20] that the European option on a zero coupon has a Black-Scholes-like formula with volatility given by q.

For the two-factor HJM model given in Eq. (3.8) we have

$$q^2=\sigma_1^2(T-t_*)^2(t_*-t_0)+\frac{\sigma_2^2}{2\lambda^3}(1-e^{-\lambda(T-t_*)})^2 \times(1-e^{-2\lambda(t_*-t_0)}). \tag{5.15}$$

A *cap* is a financial instrument for reducing ones exposure to interest rate fluctuations, and guarantees a maximum interest rate for borrowings over a fixed time. A *cap* fixes the maximum interest for a fixed period from t_* to t_*+T; the buyer of the instrument then pays for this period the maximum of the London interbank offered rate $L\equiv L(t_*,t_*+T)$ *or* the cap rate K. The cap is exercised at time t_* and the payments are made, in arrears, at time t_*+T. Let the principal amount be V; the value of the cap $\mathcal{C}$ at time t_* is then given by

$$\mathcal{C}(t_*,t_*,T)=VT(L-K)_+. \tag{5.16}$$

We have in terms of the forward rates [9]

$$TL(t_*,t_*+T)=\exp\left[\int_{t_*}^{T+t_*}dxf(t_*,x)\right]-1, \tag{5.17}$$

$$=\frac{1}{P(t_*,t_*+T)}-1. \tag{5.18}$$

The price of the $\mathcal{C}$ at time $t_0<t_*$ is given by

$$\mathcal{C}(t_0,t_*,T,K)=VE_{[t_0,t_*]}\left[\exp\left\{-\int_{t_0}^{t_*}dtf(t,t)\right\}\times\left(\frac{1}{P(t_*,t_*+T)}-1-TK\right)_+\right], \tag{5.19}$$

$$=V\int_{-\infty}^{+\infty}dH\Gamma(H)(e^H-1-TK)_+. \tag{5.20}$$

Carrying out an analysis similar to the one done for the pricing of the European call option we obtain, as in Eq. (5.14) (note the minus sign of H)

$$\Gamma(H)=\Psi(-H,t_*,T+t_*) \tag{5.21}$$

$$=\sqrt{\frac{1}{2\pi q_{cap}^2}}\exp\left\{-\frac{1}{2q_{cap}^2}\left[-H+\int_{t_*}^{T+t_*}dx \times f(t_0,x)-\frac{q_{cap}^2}{2}\right]^2\right\} \tag{5.22}$$

with q_{cap} for the two-factor model given, similar to Eq. (5.15), by

$$q_{cap}^2=\sigma_1^2T^2(t_*-t_0)+\frac{\sigma_2^2}{2\lambda^3}(1-e^{-\lambda T})^2(1-e^{-2\lambda(t_*-t_0)}). \tag{5.23}$$

The above formula shows that a cap is equivalent to a European put option on the zero-coupon bond $P(t_*,t_*+T)$. For a *caplet* the time T is taken to be small so that $L(t_*,t_*+T) \simeq f(t_*,t_*)$, and in Eq. (5.22) we have

$$\int_{t_*}^{T+t_*} dx f(t_0,x) \simeq T f(t_0,t_*), \tag{5.24}$$

$$q_{cap}^2 \simeq T^2\left[\sigma_1^2(t_*-t_0) + \frac{\sigma_2^2}{2\lambda}(1-e^{-2\lambda(t_*-t_0)})\right]. \tag{5.25}$$

VI. QUANTUM MECHANICS AND QUANTUM FIELD THEORY

Since some readers may not be familiar with the concept of a *quantum field*, the following is a brief description. We first discuss the case of quantum mechanics, and then show how the concept of a quantum field is a natural generalization of quantum mechanics.

Consider the position of a particle $x(t)$ as a function of time t. Let the particle be experimentally observed to be at position x_1 at time t_1 and at position x_2 at time t_2. How does one describe the evolution of the particle in quantum mechanics? As is well known, there are three independent and equivalent descriptions of quantum mechanics. We discuss quantum evolution in Euclidean time, as this makes the connection to finance more straightforward.

In the Schrodinger representation of quantum mechanics [21], the probability for the particle to be found in the interval $[x,x+dx]$ for any time $t \in [t_1,t_2]$ is given by $|\psi(x,t)|^2 dx$. In other words, the probability for the particle to be found, at time t, in the neighborhood of point x_0 is given by

$$P(x_0 < x < x_0+dx) = |\psi(x_0,t)|^2 dx. \tag{6.1}$$

The complex valued function $\psi(x,t)$ is the Schrödinger wave function.

The second quantum description is provided by the Heisenberg operator equations [21] in which the position, momentum, energy, and so on of the particle are considered to be noncommuting operators acting on the Hilbert space of physical states. The probability amplitude for making a transition from its initial position to its final position is given by the square of the absolute value of the transition amplitude, namely, $|\langle x_2|e^{-\tau\hat{H}}|x_1\rangle|^2$, where $\tau = t_2 - t_1$, and $\hat{H}$ is the Hamiltonian operator driving the evolution of the quantum particle.

The third description of the quantum particle is given by the Feynman path integral [3,7]. In this representation, the position of the particle $x(t)$ at each instant $t \in [t_1,t_2]$ is considered to be an *independent random variable*. The transition amplitude $\langle x_2|e^{-\tau\hat{H}}|x_1\rangle$ is obtained by integrating over possible values of all the random variables $x(t)$, and yields the Feynman path integral [3,7] given by

$$\langle x_2|e^{-\tau\hat{H}}|x_1\rangle = \prod_{t_1<t<t_2} \int_{-\infty}^{+\infty} dx(t) e^S \tag{6.2}$$

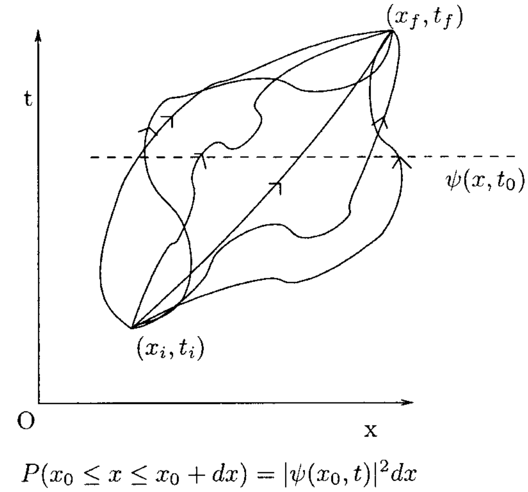

FIG. 6. Quantum particle simultaneously taking all possible virtual paths from (x_i,t_i) to (x_f,t_f).

with the boundary condition $x(t_1)=x_1$ and $x(t_2)=x_2$. The (functional) integration given in Eq. (6.2) can be thought of as summing over *all possible paths* that the quantum particle can take between points x_1 and x_2, and hence the term path integration. One can think of the quantum particle *simultaneously* taking, in a virtual sense, all the possible paths from x_1 to x_2.

The quantum virtual paths are figuratively shown in Fig. 6, together with the interpretation of the Schrodinger wave function as determining the probability for the particle to found near some point x.

The ''action'' S is a functional of the paths, and can be constructed from the Hamiltonian $\hat{H}$. A general form for the action is given by

$$S = -\frac{1}{2}\int_{t_1}^{t_2} dt\left[\left(\frac{dx}{dt}\right)^2 + V(x)\right], \tag{6.3}$$

where $V(x)$ is the potential that the particle is moving in.

The formalism of quantum mechanics is based on conventional mathematics of partial differential equations and functional analysis. The infinite dimensional integration measure given by $\Pi_{t_1<t<t_2}\int_{-\infty}^{+\infty} dx(t)$ can be given a rigorous, measure theoretic, definition as the integration over all continuous, but nowhere differentiable, paths running between points x_1 and x_2.

In Feynman's representation of quantum mechanics, one computes the *matrix elements* of operators using functional integration, and hence the structure of the Hilbert space of states, and the noncommuting operator algebra acting on this space, and so on, is present only in an implicit manner.

We have discussed the quantum mechanical evolution of a single particle, and it is not too difficult to see that the formalism extends without much change to that of N particles.

Suppose we are interested in studying how an extended object, say a *string*, undergoes quantum evolution? How do

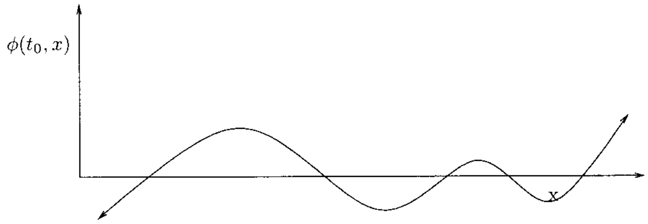

FIG. 7. A typical string configuration.

we describe the quantum dynamics of such an object? The formalism of quantum field theory [7] has been developed to answer this question.

For simplicity, consider a nonrelativistic (one dimensional) string, and let its displacement from equilibrium at time t and at position x be denoted by $\phi(t,x)$, as shown in Fig. 7 for a particular instant t_0. The string's position is called a *field*, in this case, the string field.

Let the initial string position at time t_1 be given by $\phi_1(x)=\phi(t_1,x)$, and the final position at time t_2 be given by $\phi_2(x)=\phi(t_2,x)$. Suppose the string has mass per unit length given by ρ, and string tension (energy per unit length) given by T. A general expression for the action of the string, namely, S_{string} is given by [7]

$$\begin{aligned} S_{\text{string}} &= -\frac{1}{2}\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \rho \left(\frac{\partial \phi}{\partial t}\right)^2 \\ &\quad -\frac{1}{2}\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx \left[T\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi)\right] \\ &\equiv S_{\text{kinetic}} + S_{\text{potential}}, \end{aligned} \tag{6.4}$$

where, as expected, $V(\phi)$ is the potential of the field ϕ. In analogy with quantum mechanics, we allow for *all possible string positions* to occur at each instant of the string's evolution. Hence we need to integrate over all possible values for the string's position at each point x and for each instant t.

Let the dynamics of the field be determined by the Hamiltonian of the string given by $\hat{H}_{\text{string}}$, and which can be derived from the string action S_{string}. The initial and final quantum state vectors of the (string) field is given by $|\phi_1\rangle = \otimes_{-\infty<x<+\infty}|\phi(x)\rangle$ and $|\phi_2\rangle = \otimes_{-\infty<x<+\infty}|\phi'(x)\rangle$, respectively.

We hence have, in analogy with quantum mechanics, that the quantum field theory of the string field $\phi(t,x)$ is defined by the Feynman path integral, and yields the transition amplitude [7]

$$Z \equiv \langle \phi_2 | e^{-\tau \hat{H}_{\text{string}}} | \phi_1 \rangle, \tag{6.5}$$

$$= \prod_{t_1<t<t_2} \prod_{-\infty<x<+\infty} \int_{-\infty}^{+\infty} d\phi(t,x) \exp(S_{\text{string}}) \tag{6.6}$$

with boundary conditions given by $\phi_1(x)=\phi(t_1,x)$ and $\phi_2(x)=\phi(t_2,x)$. The object $\{\phi(t,x)\}$ is called a *quantum field,* since — unlike a classical string which has a determinate and fixed value for every x and t — the quantum field takes *all possible values* for each x and t.

Equations (6.2) and (6.5) that define quantum mechanics and quantum field theory, respectively, look deceptively similar. In quantum mechanics, at any instant t_0 there is only one random variable $x(t_0)$, whereas for a quantum field, there are *infinitely many* random variables, since for *each x* the string coordinate $\phi(t_0,x)$ is an independent random variable. More precisely, for a single particle the Hilbert space of states for quantum mechanics depends on one variable, namely is given by $|x\rangle$, whereas for a single field ϕ, the field's Hilbert space depends on infinitely many independent variables given by the infinite tensor product $|\phi\rangle = \otimes_{-\infty<x<+\infty}|\phi(x)\rangle$.

We see that quantum mechanics is a system with a finite number of random variables, whereas a field theory has infinitely many independent random variables. This, in essence, is the difference between quantum mechanics and quantum field theory.

From a more mathematical point of view there is no measure theoretic interpretation of the expression $\Pi_{t_1<t<t_2}\Pi_{-\infty<x<+\infty}\int_{-\infty}^{+\infty}d\phi(t,x)$. The only rigorous definition of Eq. (6.6) is to limit the volume of spacetime to be finite, and then discretize space time so that the infinite dimensional integration given in Eq. (6.6) is reduced to an ordinary finite dimensional multiple integral. A (finite) continuum limit of a nonlinear field theory defined on a finite and discrete spacetime is in general possible only if the action S defines a theory that is renormalizable. The procedure of renormalization has as yet no mathematically rigorous definition, and, in general, the entire formalism of quantum field theory lies beyond the scope of conventional and rigorous mathematics.

If the action S is only a quadratic function of the quantum field ϕ, the theory is said to be a free field (Gaussian), and one can take the continuum limit without having to address the problem of renormalization. Fortunately, for the model proposed for the forward rates, the action will be quadratic.

VII. THE ACTION FOR THE FORWARD RATES

As mentioned earlier, in the HJM model the fluctuations in the forward rates at a given time t are given by ''shocks'' $W_i(t)$ that are delivered to the entire curve $f(t,x)$ by random variables that do not depend on the maturity direction x. Clearly, a more general evolution of the instantaneous forward rate would be to let the whole curve evolve randomly, that is to let *all the forward rates* — that is, $f(t,x)$ for *each x and t*, fluctuate *independently*. The only constraint imposed on the random evolution of the forward rates is that, at every instant, the condition of *no arbitrage* be valid.

At any instant t, there exist in the market forward rates for a duration of T_{FR} in the future; so, for example, if t refers to present time t_0, then one has forward rates from t_0 till time t_0+T_{FR} in the future. In the market, T_{FR} is at least about 30 years, and hence we have $T_{FR}>30$ years. In general, at any time t, all the forward rates exist till time $t+T_{FR}$ [22]. The forward rates at any instant t is denoted by $f(t,x)$, with $t<x<t+T_{FR}$, and is called the *forward rate curve.*

Since at any instant t there are infinitely many forward rates, we need an infinite number of independent variables to

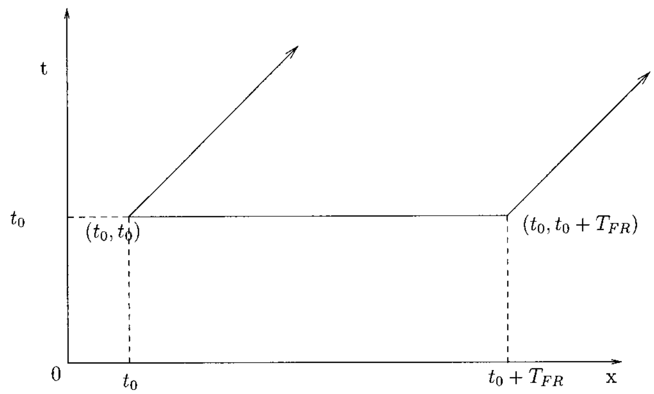

FIG. 8. Domain of the forward rates.

describe its random evolution. As discussed in Sec. VI, the generic quantity describing such a system is a quantum field [7]. For modeling the forward rates and treasury bonds, we consequently need to study a two-dimensional quantum field on a finite Euclidean domain.

We hence consider the forward rates to be a *quantum field*; that is, $f(t,x)$ is taken to be an *independent* random variable for each x and each t. For notational simplicity we keep both t and x continuous; in Appendix A, a case with both t and x discrete is analyzed and the continuum limit discussed in some detail.

For the sake of concreteness, consider the forward rates starting from some time t_0 to the infinite future, that is, with $t=\infty$. Since all the forward rates $f(t,x)$ are always for the future, we have $x>t$; hence the quantum field $f(t,x)$ is defined on the domain in the shape of a semi-infinite parallelogram $\mathcal{P}$ that is bounded by parallel lines $x=t$ and $x=T_{FR}+t$ in the maturity direction, and by the line $t=T_0$ in the time direction, as shown in Fig. 8. Every point in the domain $\mathcal{P}$ represents an independent integration variable $f(t,x)$, and shows the enormous increase over the HJM random variable $W_i(t)$ given in Fig. 1.

To define an action for the forward rates, we first need a kinetic term, as is common to all field theories, and which, for example, is given in Eq. (6.4) by S_{kinetic}. Since we know from the HJM model that the forward rates have a drift velocity $\alpha(t,x)$ and volatility $\sigma(t,x)$, these have to appear directly in the action for the forward rates.

The analog of a potential for the forward rates is to let the forward rates store ''energy'' in the shape of the curve; to represent this property of the forward rates considered as a string, we introduce a new parameter μ, which is the analog of string tension and which quantifies the strength of the fluctuations of the forward rates in the time-to-maturity direction x. We expect that, in the limit of $\mu\rightarrow 0$, we should recover the HJM model. The simplest term that can control the fluctuations in the x direction is the gradient of $f(t,x)$ with respect to x. The action for the forward rates that generalizes the HJM action of Eq. (2.10), is given by

$$S[f]=\int_{t_0}^{\infty}dt\int_{t}^{t+T_{FR}}dx\mathcal{L}[f]\equiv\int_{\mathcal{P}}\mathcal{L}[f] \tag{7.1}$$

with the Lagrangian $\mathcal{L}[f]$ given by

$$\mathcal{L}[f]=\mathcal{L}_{\text{kinetic}}[f]+\mathcal{L}_{\text{potential}}[f], \tag{7.2}$$

$$=-\frac{1}{2T_{FR}}\left[\left\{\frac{\frac{\partial f(t,x)}{\partial t}-\alpha(t,x)}{\sigma(t,x)}\right\}^2+\frac{1}{\mu^2}\left\{\frac{\partial}{\partial x}\left(\frac{\frac{\partial f(t,x)}{\partial t}-\alpha(t,x)}{\sigma(t,x)}\right)\right\}^2\right]. \tag{7.3}$$

The initial condition is given by

$$t=t_0,\quad t_0<x<t_0+T_{FR}:$$

$$f(t_0,x):\quad \text{initial forward rate curve} \tag{7.4}$$

and the field $f(t,x)$ on the rest of the boundary points of the semi-infinite parallelogram $\mathcal{P}$ are integration variables.

Comparing Eqs. (6.4) and (7.2), we see that the forward rates behave like a (quantum) string, with a time and space dependent drift velocity $\alpha(t,x)$, an effective mass given by $1/\sigma(t,x)$, and string tension proportional to $1/\mu^2$.

The quantum field theory is defined by integrating over all configurations of $f(t,x)$ and yields

$$Z=\int Dfe^{S[f]}, \tag{7.5}$$

$$\int Df=\prod_{(t,x)\epsilon\mathcal{P}}\int_{-\infty}^{+\infty}df(t,x). \tag{7.6}$$

Note that $e^{S[f]}/$ Z is the probability for different field configurations to occur when the functional integral over $f(t,x)$ is performed.

The presence of the second term in the action given in Eq. (7.1) seems to be justified from the phenomenology of the forward rates [23] and is not ruled out by no arbitrage.

The action given in Eq. (7.1) is suitable for studying the formal properties of the forward rates. However it is often simpler for computational purposes to change variables. Let $A(t,x)$ be a two-dimensional quantum field; we use the HJM change of variables to express $A(t,x)$ in terms of the forward rates $f(t,x)$, namely;

$$\frac{\partial f}{\partial t}(t,x)=\alpha(t,x)+\sigma(t,x)A(t,x). \tag{7.7}$$

The Jacobian of the above transformation is a constant and hence, up to a constant

$$\int Df\rightarrow\int DA. \tag{7.8}$$

The quantum field theory is defined by a functional integral over all variables $A(t,x)$; the values of $A(t,x)$ on the boundary of $\mathcal{P}$ are integration variables; this yields the partition function

$$Z = \int DAe^{S[A]}, \tag{7.9}$$

where the action in terms of the $A(t,x)$ field is given by

$$S[A] = -\frac{1}{2T_{FR}}\int_{t_0}^{\infty}dt\int_t^{t+T_{FR}}dx\left\{A^2(t,x) + \frac{1}{\mu^2}\left(\frac{\partial A(t,x)}{\partial x}\right)^2\right\} \tag{7.10}$$

$$= \int_{\mathcal{P}}\mathcal{L}[A]. \tag{7.11}$$

Note that Eqs. (7.7) and (7.10) can easily be generalized to the K-factor case by introducing K-independent and identical quantum fields $A_i(t,x)$. The forward rates are then defined by the equation

$$\frac{\partial f}{\partial t}(t,x) = \alpha(t,x) + \sum_{i=1}^{K}\sigma_i(t,x)A_i(t,x). \tag{7.12}$$

For simplicity, we will only analyze the one-factor model.

It is shown in Eq. (A16) that if we define

$$W(t) = \int_t^{t+T_{FR}}dxA(t,x), \tag{7.13}$$

then, for $\mu\to 0$, we have

$$S[A]\to S_0 = -\frac{1}{2}\int_{t_0}^{\infty}dtW^2(t), \tag{7.14}$$

$$\int DA \to \int DW. \tag{7.15}$$

From Eq. (2.10) and above we see that we recover the HJM model in the $\mu\to 0$ limit. We see from Eq. (7.13) that the HJM model is a drastic truncation of the full field theory, and only considers the fluctuations of the average value of the quantum field $A(t,x)$; it in effect ''freezes-out'' all the other fluctuations of $A(t,x)$.

If one thinks of the field $A(t_0,x)$ at some instant t_0 as giving the position of a ''string'' [8,9], then in the HJM model this string is taken to be a *rigid* string. The action $S[A]$ given in Eq. (7.10) allows *all* the degrees of freedom of the field $A(t_0,x)$ to fluctuate independently and can be thought of as a ''string'' with string tension equal to $1/\mu^2$; in this language the HJM model considers the forward rate curve to be a string with infinite tension and hence rigid.

VIII. PROPAGATOR OF THE FORWARD RATES

The moment generating functional for the quantum field theory is given by the Feynman path integral as

$$Z[J] = \frac{1}{Z}\int DA\exp\left\{\int_{t_0}^{\infty}dt\int_t^{t+T_{FR}}dxJ(t,x)A(t,x)\right\}e^{S[A]}. \tag{8.1}$$

We evaluate $Z[J]$ exactly in Appendix B, and from Eq. (B17)

$$Z[J] = \exp\frac{1}{2}\int_{t_0}^{\infty}dt\int_t^{t+T_{FR}}dxdx'J(t,x) \times D(x,x';t,T_{FR})J(t,x'). \tag{8.2}$$

The propagator $D(x,x';t,T_{FR})$ is given, for $\lambda = x-t, \lambda' = x'-t$, from Eq. (B19) by

$$\begin{aligned} D(x,x';t,T_{FR}) &= \frac{\mu T_{FR}}{\sinh(\mu T_{FR})}\Bigg[\sinh\mu(T_{FR}-\lambda)\sinh(\mu\lambda')\theta(\lambda-\lambda') \\ &+ \sinh\mu(T_{FR}-\lambda')\sinh(\mu\lambda)\theta(\lambda'-\lambda) \\ &+ \frac{1}{2\cosh^2\left(\frac{\mu T_{FR}}{2}\right)}\left\{2\cosh\mu\left(\lambda-\frac{T_{FR}}{2}\right)\right. \\ &\times\cosh\mu\left(\lambda'-\frac{T_{FR}}{2}\right) + \sinh(\mu\lambda)\sinh(\mu\lambda') \\ &\left.+ \sinh\mu(T_{FR}-\lambda)\sinh\mu(T_{FR}-\lambda')\right\}\Bigg]. \end{aligned} \tag{8.3}$$

Note the important property of the propagator $D(x,x';t,T_{FR})$ that it depends *only* on the variables λ and λ'. This property of the propagator implies that all the properties of the future rates depend only on the how long in the future we are looking at, and not at what instant *t*.

To understand the significance of the propagator $D(x,x';t,T)$ note that the correlator of the field $A(t,x)$, for $t_0 < t,t' < t_0 + T_{FR}$, is given by

$$E(A(t,x)A(t',x')) = \frac{1}{Z}\int DAe^{S[A]}A(t,x)A(t',x') \tag{8.4}$$

$$= \delta(t-t')D(x,x';t,T_{FR}). \tag{8.5}$$

In other words, $D(x,x';t,T)$ is a measure of the effect the value of field $A(t,x)$ at maturity time x has on $A(t,x')$, namely, on its value at another maturity x'.

Since $D(x,x';t,T_{FR})$ looks fairly complicated, we examine it in a few extreme limits, and obtain

$$D(x,x';t,T_{FR}) \simeq \begin{cases} 1 + O(\mu^2), & \mu\to 0 \\ \frac{1}{2}\mu T_{FR}e^{-\mu|x-x'|}\to T_{FR}\delta(x-x'), & \mu\to\infty. \end{cases} \tag{8.6}$$

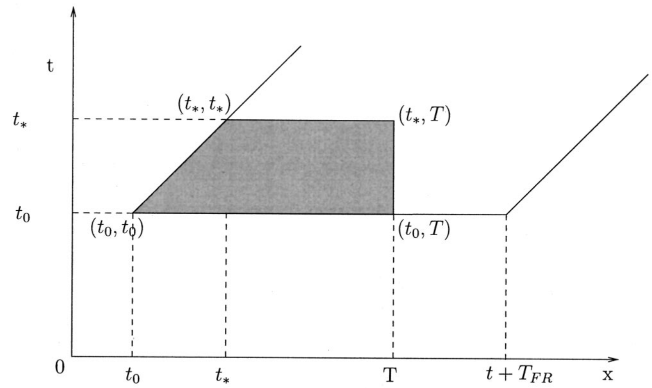

FIG. 9. Domain for no arbitrage contained in the domain of the forward rates.

We see that, as expected, in the limit of $\mu\rightarrow 0$ all the fluctuations in the x direction are "frozen," that is, are exactly correlated; in other words the values of $A(t,x)$ for different maturities are all the same, and this is the limit of the HJM model.

The propagator above has a simple interpretation for the case of $\mu\rightarrow\infty$. If the field $A(t,x)$ has some value at point x, then the field at "distances" $x-\mu^{-1}<x'<x+\mu^{-1}$ will tend to have the same value, whereas for other values of x' the field will have arbitrary values. Hence we see that in the limit of $\mu\rightarrow 0$ the fluctuations in the time-to-maturity x direction are strongly correlated within maturity time μ^{-1}, which is the *correlation time* of the forward rates.

Define

$$j(t)=\int_t^{t+T_{FR}} dx J(t,x). \tag{8.7}$$

We have from Eqs. (8.2), (8.6), and (8.7) that

$$\lim_{\mu\rightarrow 0} Z[j]=\exp\frac{1}{2}\int_{t_0}^{\infty} dt j^2(t), \tag{8.8}$$

which is the result obtained earlier in Eq. (2.12).

IX. CONDITION OF NO ARBITRAGE

We now derive the no-arbitrage condition for the action $S[A]$. Equation (3.2) for the Martingale is unchanged; generalizing Eqs. (3.3) and (3.5) we have

$$\exp\int_{\mathcal{T}}\alpha(t,x)=\frac{1}{Z}\int DA\exp\left\{-\int_{\mathcal{T}}\sigma(t,x)A(t,x)\right\}e^{\int_{\mathcal{P}}\mathcal{L}[A]} \tag{9.1}$$

$$=\exp\frac{1}{2}\int_{t_0}^{t_*}dt\int_t^T dx dx'\sigma(t,x) \times D(x,x';t,T_{FR})\sigma(t,x'). \tag{9.2}$$

Note the trapezoidal domain $\mathcal{T}$ determining the condition of no arbitrage is nested inside the domain of the forward rates $\mathcal{P}$, as shown in Fig. 9.

We have

$$\int_t^T dx\alpha(t,x)=\frac{1}{2}\int_t^T dx dx'\sigma(t,x)D(x,x';t,T_{FR})\sigma(t,x'). \tag{9.3}$$

The no-arbitrage condition has to hold for any treasury bond maturing at any time $x=T$. Hence, we differentiate above expression with respect to T, and obtain the generalization of Eq. (3.7) given by

$$\alpha(t,x)=\sigma(t,x)\int_t^x dx' D(x,x';t,T_{FR})\sigma(t,x'). \tag{9.4}$$

From the empirical study of forward rate curves [23] there is evidence that the HJM model is not adequate, since no arbitrage implies that the drift term $\alpha(t,x)$ is quadratic in the volatility, and which is inconsistent with data; in [23] an additional term is added that reflects the market price of risk. In the approach of field theory, the additional terms due to the propagator could provide a better model of no arbitrage for the drift term.

We have the following limiting behavior

$$\alpha(t,x)\simeq\begin{cases}\sigma(t,x)\int_t^x dx'\sigma(t,x'), & \mu\rightarrow 0\\ \frac{1}{2}T_{FR}\sigma^2(t,x), & \mu\rightarrow.\infty\end{cases} \tag{9.5}$$

Note the equation for $\alpha(t,x)$ given above that the case for $\mu=\infty$ is quite dis-similar from that of the HJM model given in Eq. (3.7), which is the case for $\mu=0$. The expression for $\alpha(t,x)$ given in Eq. (9.4) for $\mu\neq 0$ continuously interpolates between the extreme values of $\mu=0$ and $\mu=\infty$.

For the case of the one-factor model we have the exact result that

$$\alpha(t,x)=\frac{\sigma_1^2 T_{FR}}{2}\left[1+\frac{\exp[-2\mu(T_{FR}-x+t)]-e^{-2\mu(x-t)}}{1-e^{-2\mu T_{FR}}}\right]. \tag{9.6}$$

The limiting behavior for the one-factor model, which also directly follows from Eq. (9.5), is given by

$$\alpha(t,x)\simeq\begin{cases}\sigma_1^2(x-t) & \mu\rightarrow 0\\ \frac{1}{2}\sigma_1^2 T_{FR} & \mu\rightarrow\infty.\end{cases} \tag{9.7}$$

Note that in the limit of $\mu\rightarrow 0$, we recover the HJM result.

We have from Eqs. (7.7) and (9.4) that

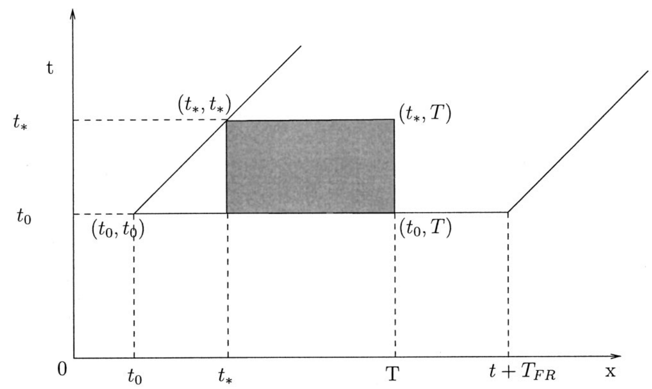

FIG. 10. Domain for the futures price.

$$f(t,x)=f(t_0,x)+\int_{t_0}^{t}dt'\sigma_i(t',x)\times\int_{t'}^{x}dyD(x,y;t',T_{FR})\sigma_i(t',y)+\int_{t_0}^{t}dt'\sigma_i(t',x)A_i(t',x). \quad (9.8)$$

X. FUTURES AND OPTION PRICING

We derive the futures and options pricing using quantum field theory. For the two-factor model all the expressions can be obtained exactly; the results for the $\mu=0$ limit are the same as the HJM model; we will explicitly give the results only the one-factor model.

Equation (4.7) for the futures price $\mathcal{F}$ for the case of the field theory model for the forward rates only changes the formula for Ω. From Eq. (4.8), and for the domains $\mathcal{R}$ and $\mathcal{P}$ given in equation below — and as shown in Fig. 10 — we have

$$e^{\Omega}=\frac{1}{Z}\int DA\exp\left\{-\int_{\mathcal{R}}dx\sigma(t,x)A(t,x)e^{\int_{\mathcal{P}}\mathcal{L}[A]}\right\}, \quad (10.1)$$

$$=\exp\left\{\frac{1}{2}\int_{t_0}^{t_*}dt\int_{t_*}^{T}dxdx'\sigma(t,x)D(x,x';t,T_{FR})\sigma(t,x')\right\}. \quad (10.2)$$

Using the no-arbitrage condition (9.4) we obtain the generalization of Eq. (4.10) as given by

$$\Omega_{\mathcal{F}}(t_0,t_*,T)=-\sum_{i=1}^{K}\int_{t_0}^{t_*}dt\int_{t}^{t_*}dx\sigma_i(t,x)\int_{t_*}^{T}dx'\times D(x,x';t,T_{FR})\sigma_i(t,x'). \quad (10.3)$$

For the one-factor model, with $\sigma_1\neq 0$, we have, for $T_{FR}\geqslant T$,

$$\Omega_{\mathcal{F}}(t_0,t_*,T)=-\frac{\sigma_1^2T_{FR}}{4\mu^2}(1-e^{-\mu(T-t_*)})\times\{2\mu(t_*-t_0)+e^{-2\mu(t_*-t_0)}-1\}. \quad (10.4)$$

For the price of a European call option C, a calculation similar to the one carried out in Sec. IV gives the same formula for $\Psi(G)$ given in Eq. (5.14) with q^2 given in Eq. (5.12) replaced by

$$q^2=\sum_{i=1}^{K}\int_{t_0}^{t_*}dt\int_{t_*}^{T}dxdx'\sigma_i(t,x)D(x,x';t,T_{FR})\sigma_i(t,x'). \quad (10.5)$$

We have

$$\lim_{\mu\to\infty}q^2=T_{FR}\sum_{i=1}^{K}\int_{t_0}^{t_*}dt\int_{t_*}^{T}dx\sigma_i^2(x,t). \quad (10.6)$$

For the one-factor model we have, for $T_{FR}\geqslant T$, that

$$q^2=\frac{\sigma_1^2T_{FR}}{4\mu^2}[4\mu(t_*-t_0)\{\mu(T-t_*)+e^{-\mu(T-t_*)}-1\}+e^{-2\mu(T-t_*)}-e^{-2\mu(T-t_0)}+2(1-e^{-\mu(T-t_*)})\times(1-e^{-2\mu(t_*-t_0)})]. \quad (10.7)$$

Note for both the futures and option prices, the presence of μ is like adding another factor to the model.

If we are interested in pricing any path dependent option or other derivatives, it is not sufficient to know only the propagator $D(x,x';t,T_{FR})$; the full structure of the action $S[A]$ is then required.

For example, the payoff function of an Asian option at time t_0 on a zero-coupon bond $P(t,T)$ with exercise time t_* is given by

$$g[P(*,T)]=\left[\frac{1}{t_*-t_0}\int_{t_0}^{t_*}dtP(t,T)-K\right]_+. \quad (10.8)$$

Another example is the price of a European call option on a coupon bond $\mathcal{B}(t,T)$ given in Eq. (2.1); the payoff function is given by

$$g[\mathcal{B}]=[\mathcal{B}(t_*,T)-K]_+. \quad (10.9)$$

The payoff function $g[A]$ in both the cases above is path dependent. Expressing all the zero-coupon bonds in terms of the quantum field $A(t,x)$, the prices of such path dependent options at time t_0 are given by

$$C(t_0,t_*,T,K)=\frac{1}{Z}\int DA\exp\left[-\int_{t_0}^{t_*}dtr(t)\right]g[A]e^{S[A]}. \quad (10.10)$$

The computation above can only be performed numerically [24]; for this the functional integral over $A(t,x)$ has to be discretized, which is briefly discussed in Appendix A.

XI. CONCLUSIONS

We have reformulated the theory of treasury bonds in terms of path integration. The HJM model has a simple path integral realization with an ultralocal action. Equations for the no-arbitrage condition as well as the evaluation of futures and options were shown to be calculable in a straightforward manner using path integration. The motivation for rederiving the well-known results of the HJM model was first to understand the path integral formulation of the quantities of interest in finance, and second to generalize these quantities to the case of quantum field theory.

The quantum field theory of treasury bonds is more general than the HJM model; in particular, the correlation of fluctuations of the forward rates can be easily modeled to be finite in the field theory whereas in the HJM model *all* the fluctuations are exactly correlated. From the point of view of finance, it is unreasonable to assume that the all forward rates fluctuate identically as in the HJM-model. The multifactors in HJM model are an attempt to model the finite correlation in the time to maturity that should exist for the forward rates, and that is more efficiently captured using a finite tension in the field theory model.

We considered a Gaussian model for the field theory generalization of the HJM model as this is the simplest extension, and also because the model could be solved exactly. In particular, the formulas for the futures, cap and option price of treasury bonds were derived and involved nontrivial correlations in the volatility of the model.

We can generalize the model to account for stochastic volatility of the forward rates. This entails introducing another quantum field for modeling the fluctuations of volatility, and is similar to the quantum mechanical treatment of volatility for a single security [6]. Stochastic volatility makes the system highly nonlinear. The best way of modeling treasury bonds in practice is a computational and empirical question [25,26]. A detailed empirical study of the model for the treasury bonds and forward rates proposed in this paper, including calibration and consistency checks, is treated in some detail in [27].

For the more theoretical aspects of finance, the methodology of field theory certainly adds to the ways of studying and understanding the stochastic processes that drive the capitals market.

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APPENDIX A: LATTICE FORMULATION

We present a rigorous treatment of the quantum field theory of the forward rates. The main idea is to truncate the full functional integral given in Eq. (6.6) into a finite dimensional multiple integral by replacing the continuous domain $\mathcal{P}$ by a finite set of lattice points in a discrete domain $\hat{\mathcal{P}}$. The way this is done is to discretize the continuous plane xt into a finite lattice. One can then discuss more rigorously the continuum limit as the limit of the lattice theory.

We discretize the domain $\mathcal{P}$ into a lattice of discrete points. Let $(t,x)\to(m\epsilon,na)$, where ϵ is an infinitesimal time step and a is an infinitesimal in the x direction. Truncate the semi-infinite domain $\mathcal{P}$ given in Fig. 8 into a finite discretize domain $\hat{\mathcal{P}}$, with an upper limit in the time direction given by $M\epsilon$. Let $N=T_{FR}/a$ and $m_0=t_0/\epsilon$.

The discrete and finite domain $\hat{\mathcal{P}}$ is bounded in the time direction by $m=m_0$ and $m=M$, and in the maturity direction by $m\epsilon=na$ and $na=m\epsilon+Na$. The integers take values in the discrete domain $\hat{\mathcal{P}}$, and are given by

$$\hat{\mathcal{P}}=\{m=m_0,m_0+1,\ldots M-1;\quad na=m\epsilon,m\epsilon+a,\ldots m\epsilon+Na\}. \tag{A1}$$

The forward rates and quantum field yield on discretization

$$f(t,x)\to f(m\epsilon,na)\equiv f_{mn}, \tag{A2}$$

$$A(t,x)\to A(m\epsilon,na)\equiv A_{mn}, \tag{A3}$$

and similarly for α and σ.

From Eq. (7.7) we have

$$f_{m+1n}=f_{mn}+\epsilon\alpha_{mn}+\epsilon\sigma_{mn}A_{mn}, \tag{A4}$$

$$f_{m_0,n}:n=m_0,m_0+1,\ldots,N+m_0:\quad \text{initial forward rate curve.} \tag{A5}$$

Using finite differences to discretize derivatives, we obtain from Eq. (7.10), that

$$S[A]=-\frac{\epsilon}{2(N+1)}\sum_{m=m_0}^{M-1}\left\{\sum_{n=m}^{N+m}A_{mn}^2+\frac{a}{\mu^2}\sum_{n=m}^{N+m-1}(A_{mn+1}-A_{mn})^2\right\}, \tag{A6}$$

$$\int dA=\prod_{m=m_0}^{M-1}\prod_{n=m}^{N+m}\int_{-\infty}^{+\infty}dA_{mn}. \tag{A7}$$

Note the functional integral over the field $A(t,x)$ has been reduced to a *finite-dimensional* multiple integral over the A_{mn} variables, which in the above case consists of $N(M$

$-m_0$) independent variables; hence all the techniques useful for evaluating finite-dimensional integrals can be used for performing the integration over A_{mn}.

To achieve the correct normalization, one in fact need not keep track of the constants that correctly normalize $\int dA$ in Eq. (A7). Instead, one simply redefines the action by

$$e^{S[A]} \rightarrow \frac{e^{S[A]}}{Z}, \tag{A8}$$

$$Z \equiv \int dA e^{S[A]}. \tag{A9}$$

All the constants in $\int dA$ cancel out; and more importantly, the expression e^S/Z is correctly normalized to be interpreted as a probability distribution, and hence can be used for Monte Carlo studies of this theory. The action given in Eq. (A6) is the starting point for any simulations that are required of the model including the pricing of path dependent derivatives; there are well-known numerical algorithms developed in physics for numerically studying quantum fields [24].

We explicitly solve for the case of $\mu \rightarrow 0$ to see how the HJM model emerges. For $\mu \rightarrow 0$, the second term in the action gives a product of δ functions and we have

$$e^{S[A]} = e^{S_0} \prod_{m=m_0}^{M-1} \prod_{n=m}^{N+m-1} \delta(A_{mn+1} - A_{mn}), \tag{A10}$$

$$S_0 = -\frac{\epsilon}{2(N+1)} \sum_{m=m_0}^{M-1} \sum_{n=m}^{N} A_{mn}^2. \tag{A11}$$

Consider evaluating a typical expression like Z in Eq. (8.2). For each m, there are N integration variables A_{mn}; from Eq. (A10) we see that there are only $(N-1)$ δ functions, leaving, for every m, only one variable, say A_{mm} unrestricted. For simplicity, we take $\epsilon = a$; hence we have

$$Z = \prod_{m=m_0}^{M-1} \sqrt{\frac{\epsilon}{2\pi}} \int dA_{mm} e^{S_0}, \tag{A12}$$

$$S_0 = -\frac{\epsilon}{2} \sum_{m=m_0}^{M-1} A_{mm}^2. \tag{A13}$$

Defining $W(m) = A_{mm}$, we see from Eqs. (2.9) that we have recovered the HJM model. We can equivalently consider

$$W(m) = \sum_{n=m}^{N+m} A_{mn} \tag{A14}$$

and we have

$$\lim_{\mu \rightarrow 0} W(m) \rightarrow A_{mm}. \tag{A15}$$

Taking the continuum limit, we see that the field theory, in the $\mu \rightarrow 0, M \rightarrow 0$ limit reduces to

$$S_0 \rightarrow -\frac{1}{2} \int_{t_0}^{\infty} dt W^2(t), \tag{A16}$$

$$W(t) = \int_t^{t+T_{FR}} dx A(t,x). \tag{A17}$$

For the general case of $\mu \neq 0$, from Eq. (A6), taking the continuum limit of $\epsilon \rightarrow 0, a \rightarrow 0, M \rightarrow \infty$ we obtain the expected result that

$$S[A] = -\frac{1}{2T_{FR}} \int_{t_0}^{\infty} dt \int_t^{t+T_{FR}} dx \times \left\{ A^2(t,x) + \frac{1}{\mu^2} \left(\frac{\partial A(t,x)}{\partial x} \right)^2 \right\}, \tag{A18}$$

$$\int DA = \prod_{(t,x)\epsilon\mathcal{P}} \int dA \equiv \lim_{\epsilon \rightarrow 0, a \rightarrow 0, M \rightarrow \infty} \prod_{mn} \int dA_{mn}, \tag{A19}$$

$$Z = \int DA e^{S[A]}. \tag{A20}$$

APPENDIX B: GENERATING FUNCTIONAL $Z[J]$

Since the generating functional $Z[J]$ is of central importance in studying the quantum field theory, for completeness we briefly discuss its derivation; all these results are well known in physics [7] and this derivation is intended for readers from other disciplines.

Recall

$$Z[J] = \frac{1}{Z} \int DA e^{S[A,J]}, \tag{B1}$$

$$S[A,J] = \int_{t_0}^{\infty} dt \int_t^{t+T_{FR}} dx J(t,x) A(t,x) + S[A]. \tag{B2}$$

Since $S[A,J]$ is quadratic functional of the field $A(t,x)$, to perform the functional integration over the field, all we need to do is to find the specific configuration of $A(t,x)$, say $a(t,x)$ which maximizes $S[A,J]$; due to our choice of normalization $Z[J]$ depends only on $a(t,x)$.

Since there is no coupling in the time direction t, we study the solution $a(t,x)$ separately for each t, and on the finite line interval $t < x < t + T_{FR}$. We first study the case for which the boundary values of the field $A(t,x)$ are fixed, that is consider $A(t,t) = p$ and $A(t,t+T_{FR}) = p'$ to be held fixed; we will later integrate over p, p' as is required for the evaluation of $Z[J]$. We henceforth suppress the time variable t for notational convenience.

The ''classical'' (deterministic) field configuration $a(t,x) \equiv a(x)$ is defined by

$$\frac{\delta S[a,J]}{\delta A(t,x)} = 0, \tag{B3}$$

$$a(x=t)=p;a(x=t+T_{FR})=p'. \tag{B4}$$

Doing a change of variables $A(t,x)=B(t,x)+a(t,x)$ and a functional Taylors expansion we have, from Eq. (B3)

$$S[A+a,J]=S_{cl}[a,J]+\tilde{S}[B]. \tag{B5}$$

Note that due to boundary conditions given in Eq. (B4), $\tilde{S}[B]$ is independent of p,p',J. The functional integral over the $B(t,x)$ variables gives only an overall constant that can be ignored, and hence we have

$$Z[J]=\frac{1}{Z}\int_{-\infty}^{+\infty}dp\,dp'\,e^{S_{cl}[a,J]}. \tag{B6}$$

We now determine $a(x)$; from Eq. (B3) we have

$$\frac{1}{\mu^2}\frac{\partial^2 a(x)}{\partial x^2}-a(x)+T_{FR}J(x)=0, \tag{B7}$$

$$a(t)=p,a(t+T_{FR})=p';t<x<t+T_{FR}. \tag{B8}$$

Since Eq. (B7) is a linear, the solution for $a(x)$ is given by a sum of the solutions of the homogeneous and inhomogeneous equations; it can be verified that

$$a(x)=\frac{1}{\sinh(\mu T_{FR})}[a_H(x)+a_{IH}(x)] \tag{B9}$$

with the homogeneous solution given by

$$a_H(x)=p\sinh\mu(T_{FR}+t-x)+p'\sinh\mu(x-t) \tag{B10}$$

and the inhomogeneous solution given by

$$\begin{aligned}a_{IH}(x)=\mu T_{FR}\int_t^{t+T_{FR}}dx'[&\theta(x-x')\sinh\mu(T_{FR}+t-x)\\&\times\sinh\mu(x'-t)+\theta(x'-x)\sinh\mu(T_{FR}+t-x')\\&\times\sinh\mu(x-t)]J(x').\end{aligned} \tag{B11}$$

The "classical" action is given by

$$S_{cl}[a,J]=S_1[p,p';J]+S_2[J] \tag{B12}$$

with

$$\begin{aligned}S_1[p,p';J]=&-\frac{1}{2\mu\sinh(\mu T_{FR})}[\cosh(\mu T_{FR})\\&\times(p^2+p'^2)-2pp']\\&+\frac{T_{FR}}{\sinh(\mu T_{FR})}[pQ+p'P],\end{aligned} \tag{B13}$$

where

$$P=\int_t^{t+T_{FR}}dx\sinh\mu(x-t)J(x), \tag{B14}$$

$$Q=\int_t^{t+T_{FR}}dx\sinh\mu(T_{FR}+t-x)J(x), \tag{B15}$$

and

$$\begin{aligned}S_2[J]=&\frac{\mu T_{FR}^2}{\sinh(\mu T_{FR})}\int_t^{t+T_{FR}}dx\,dx'\,\theta(x-x')\\&\times\sinh\mu(T_{FR}+t-x)\sinh\mu(x'-t)J(x)J(x').\end{aligned} \tag{B16}$$

Performing the Gaussian integrations over p,p' and restoring the time variable t yields

$$Z[J]=\frac{e^{S_2[J]}}{Z}\int dp\,dp'\,e^{S_1[p,p';J]} \tag{B17}$$

$$\begin{aligned}&=\exp\frac{1}{2}\int_{t_0}^{\infty}dt\int_t^{t+T_{FR}}dx\,dx'\,J(t,x)\\&\quad\times D(x,x';t,T_{FR})J(t,x').\end{aligned} \tag{B18}$$

From Eqs. (B12), (B13), and (B16) we have, defining $\lambda=x-t$ and $\lambda'=x'-t$, and after some simplifications

$$\begin{aligned}&D(x,x';t,T_{FR})\\&=\frac{\mu T_{FR}}{\sinh(\mu T_{FR})}\Bigg[\sinh\mu(T_{FR}-\lambda)\sinh(\mu\lambda')\,\theta(x-x')\\&\quad+\sinh\mu(T_{FR}-\lambda')\sinh(\mu\lambda)\,\theta(x'-x)\\&\quad+\frac{1}{2\cosh^2\left(\frac{\mu T_{FR}}{2}\right)}\Bigg\{2\cosh\mu\left(\lambda-\frac{T_{FR}}{2}\right)\\&\quad\times\cosh\mu\left(\lambda'-\frac{T_{FR}}{2}\right)+\sinh(\mu\lambda)\sinh(\mu\lambda')\\&\quad+\sinh\mu(T_{FR}-\lambda)\sinh\mu(T_{FR}-\lambda')\Bigg\}\Bigg].\end{aligned} \tag{B19}$$

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